Q.1. State and Prove Leibnitz's Theorem

Soln: Statement: If u and v be two functions of x possessing derivatives of the nth order, then,

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n$$

Proof: Let $y = uv$

Differentiating both sides with respect to x, we get,

$$y_1 = u_1 v + u v_1$$

$$y_2 = u_2 v + u_1 v_1 + u_1 v_1 + u v_2$$
$$= u_2 v + 2u_1 v_1 + u v_2$$
$$= u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2$$

$$y_3 = u_3 v + u_2 v_1 + 2u_2 v_1 + 2u_1 v_2 + u_1 v_2 + u v_3$$
$$= u_3 v + 3u_2 v_1 + 3u_1 v_2 + u v_3$$
$$= u_3 v + {}^3C_1 u_2 v_1 + {}^3C_2 u_1 v_2 + {}^3C_3 u v_3$$

This theorem is true for $n = 1, 2, 3$.

Let $n = m$, so that, we have

$$y_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \cdots + {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r u_{m-r} v_r + \cdots + {}^mC_m u v_m$$

Diff. with respect to x on both sides, we get

$$y_{m+1} = u_{m+1} v + u_m v_1 + {}^mC_1 u_m v_1 + {}^mC_1 u_{m-1} v_2 + {}^mC_2 u_{m-1} v_2 + {}^mC_2 u_{m-2} v_2 + \cdots + {}^mC_{r-1} u_{m-r+2} v_{r-1} + {}^mC_r u_{m-r+1} v_r + {}^mC_r u_{m-r+1} v_r + {}^mC_r u_{m-r} v_{r+1} + \cdots + {}^mC_m u v_{m+1}$$

$$= u_{m+1}v + u_m v_1(1+{}^mC_1) + u_{m-1}v_2({}^mC_1+{}^mC_2) + \cdots$$

$$+ u_{m-r+1}v_r({}^mC_{r-1}+{}^mC_r) + \cdots + {}^mC_m u v_{m+1}$$

But we know that

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r,\quad 1 + {}^mC_1 = {}^{m+1}C_1, \text{ etc.}$$

$${}^mC_m = 1 = {}^{m+1}C_{m+1}$$

$$\therefore y_{m+1} = u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1}v_2 + \cdots + {}^{m+1}C_r u_{m-r+1}v_r + \cdots + {}^{m+1}C_{m+1} u v_{m+1}$$

Thus, we see that if the theorem is true for any particular value m of n, it is also true for the next higher value $m+1$ of n.

Here, it is proved to hold for 2 and 3 differentiations, hence it holds for four and so on and therefore, it is true for every positive integral value of n

Proved